$$\operatorname{arctg} 1 = \frac{\pi}{4} \Rightarrow \frac{\sqrt{e^x-4}}{2} = 1 \quad \Big| \cdot 2$$

$$\sqrt{e^x-4} = 2 \quad \Big| ^2$$

$$e^x - 4 = 4 \quad \Rightarrow \quad e^x = 8 \quad \Rightarrow \quad \boxed{x = \ln 8}$$

Za vežbu:

a) $\displaystyle\int_0^5 \frac{dx}{x^7}$

b) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$

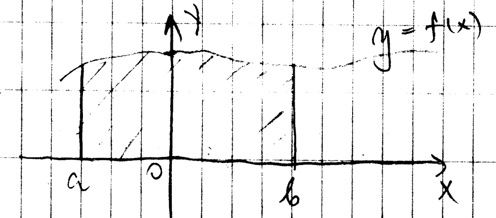
c) $\displaystyle\int_0^1 \frac{dx}{x^3-5x^2}$

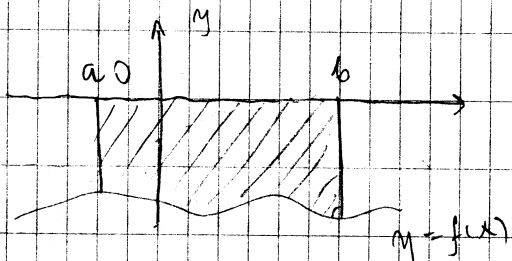
d) $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x^2-1|}}$

e) $\displaystyle\int_0^1 \frac{1}{(2-x)\sqrt{1-x}}$

20. 04. 2010

POVRŠINA RAVNOG LIKA





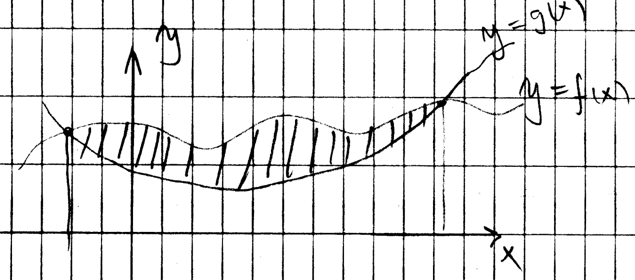
$y = f(x)$, $x = a$, $x = b$, $y = 0$

$$P = \int_a^b f(x)\,dx$$

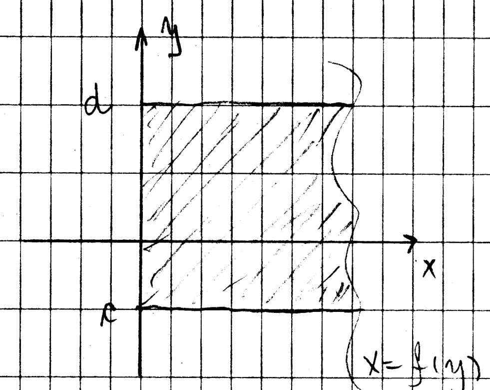
$(f(x) \geq 0)$ za $x \in [a,b]$

$f(x) \leq 0$ kad $x \in [a,b]$

$$P = \left| \int_a^b f(x)\,dx \right|$$

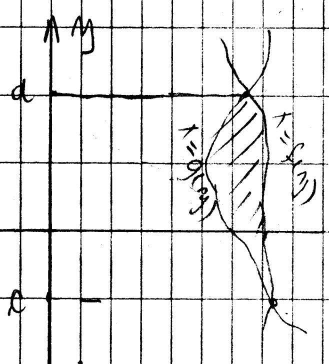


$$P=\int_a^b [f(x)-g(x)]\,dx$$



$x=f(y)$
$y=c$
$y=d$
$x=0$

$$P=\int_c^d f(y)\,dy$$



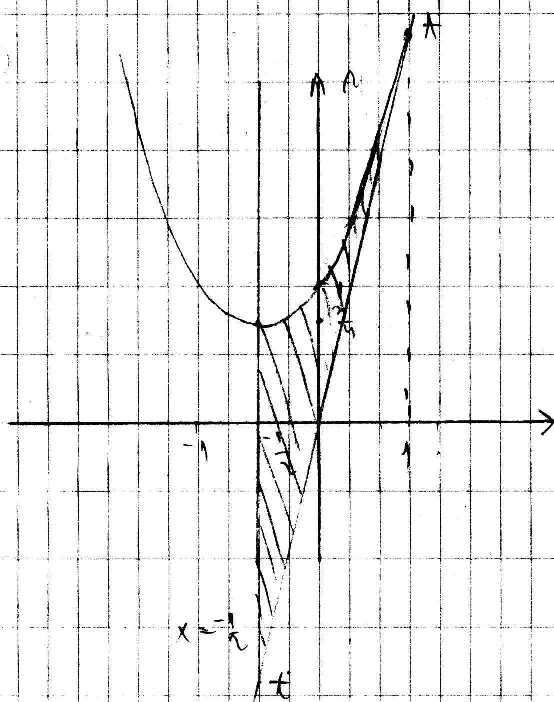
$$P=\int_c^d [f(y)-g(y)]\,dy$$

① Na parabolu $y=x^2+x+1$ povučena je tangenta u tački sa apscisom $x=1$. Izračunati površinu koju obrazuje data parabola sa povučenom tangentom i osom simetrije parabole.

$D = 1-4 = -3 \Rightarrow x_{1,2} \notin \mathbb{R}$ $\boxed{y = x^2+x+1}$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right) \Rightarrow T\left(-\frac{1}{2}, \frac{3}{4}\right)$

$y(0) = 1$



$x = 1 \Rightarrow y_1 = 3$

$A(1,3)$

$t: y - y_1 = y'(x_1)(x-x_1)$

$y' = 2x+1$

$y'(1) = 3$

$t: y-3 = 3(x-1)$

$y-3 = 3x-3$

$y = 3x$

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right) = \left(-\frac{1}{2}, \frac{3}{4}\right)$

$P = \int_{-\frac{1}{2}}^{1} (x^2+x+1-3x)\,dx = \int_{-\frac{1}{2}}^{1} (x^2-2x+1)\,dx = \dots = \frac{9}{8}$

2. U presječnim tačkama prave $x-y+1=0$ i parabole $y=x^2-4x+5$ povučene su tangente na parabolu. Izračunati površinu lika ograničenog parabolom i tangentama.

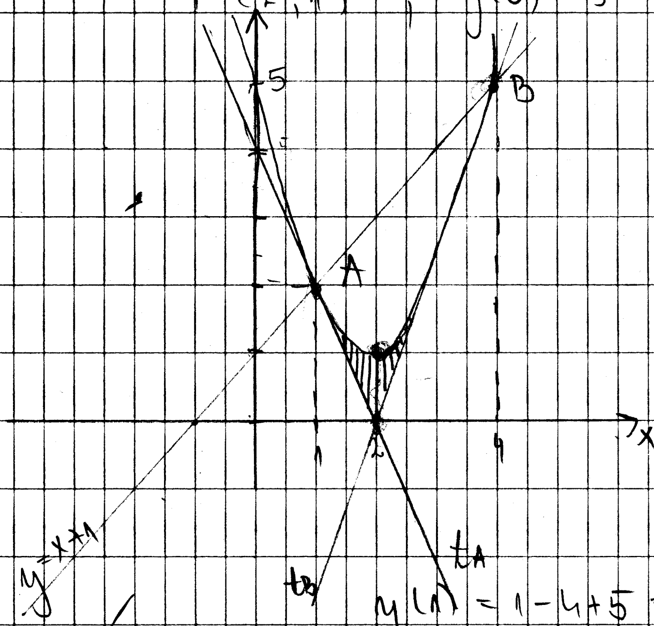
$D = 16 - 20 = -4$

$T\left(\frac{4}{2}, \frac{4}{4}\right)$

$T(2, 1)$, $y(0) = 5$

$x - y + 1 = 0 \Rightarrow x + 1 = y$

x	0	-1
$x+1$	1	0



$$\begin{cases} y = x + 1 \\ y = x^2 - 4x + 5 \end{cases}$$

$x + 1 = x^2 - 4x + 5$

$x^2 - 5x + 4 = 0$

$x_1 = 1 \qquad x_2 = 4$

$y(1) = 1 - 4 + 5 = 2$

$y(4) = 16 - 16 + 5 = 5$

$A(1, 2) \qquad B(4, 5)$

$y' = 2x - 4$

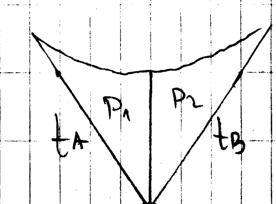
$y'(1) = 2 - 4 = -2$

$y'(4) = 8 - 4 = 4$

$t_A: \; y - 2 = -2(x - 1)$

$y - 2 = -2x + 2$

$\boxed{t_A: \; y = -2x + 4}$



$t_B: \; y - 5 = 4(x - 4)$

$\boxed{t_B: \; y = 4x - 11}$

$$\begin{cases} y = -2x + 4 \\ y = 4x - 11 \end{cases}$$

$-2x + 4 = 4x - 11$

$-6x = -15$

$x = \frac{5}{2}$

$P = P_1 + P_2$

$$P_1 = \int_1^{\frac{5}{2}} \left[x^2 - 4x + 5 - (-2x + 4)\right] dx = \int_1^{\frac{5}{2}} (x^2 - 2x + 1)\, dx \; \ldots$$

završit za vježbu!

$$P_2=\int_{\frac{5}{2}}^{4}\left[x^2-4x+5-(4x-11)\right]dx=\int_{\frac{5}{2}}^{4}(\underbrace{x^2-8x+16}_{(x-4)^2})\,dx=\left|\begin{array}{l}x-4=t\\ dx=dt\end{array}\right|\ \ldots$$

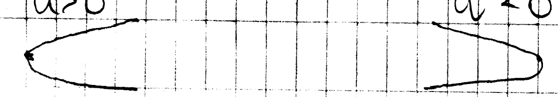
završit za vježbu

P3) Izračunati P figure koju ograničavaju linije:

a) $y^2=2x+1$, $y=2x-1$

za crtanje parabole $y^2=ax+b$

$a>0$ $\qquad$ $a<0$



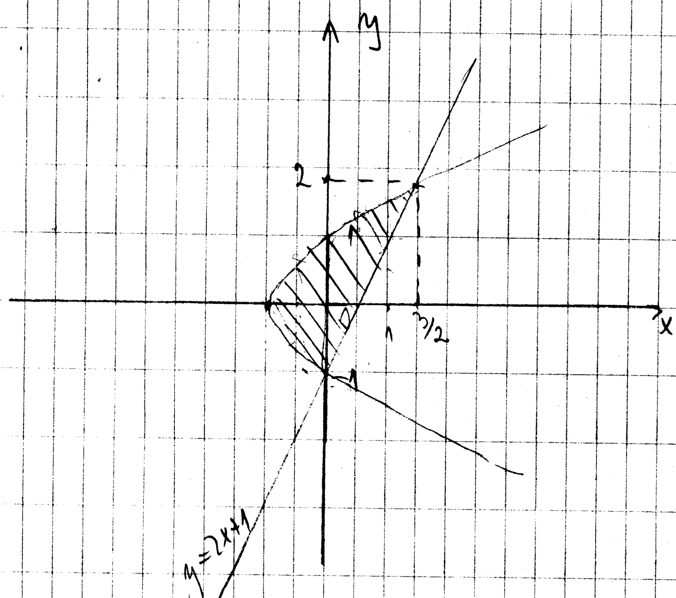
Tjeme $T(x_0,0)$

$y^2=2x+1$

$x=0 \Rightarrow y^2=1 \Rightarrow y=\pm 1$

$y=0 \Rightarrow 2x+1=0 \Rightarrow x=-\frac{1}{2}$

x	0	1
$2x-1$	-1	1



$$\begin{cases} y^2=2x+1\\ y=2x-1\end{cases}$$

$(2x-1)^2=2x+1$

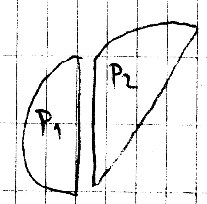
$4x^2-4x+\cancel{1}=2x+\cancel{1}$

$x_1=0 \qquad x_2=\frac{3}{2}$

$\Downarrow$

$y_1=-1 \qquad y_2=2$

I način:



$y^2=2x+1$

$y=\pm\sqrt{2x+1}$

$$P_1 = \int_{-\frac{1}{2}}^{0} \left[\sqrt{2x+1} - \left(-\sqrt{2x+1}\right)\right) dx \quad \ldots$$

$$P_2 = \int_{0}^{\frac{3}{2}} \left(-\sqrt{2x+1} - (2x-1) \right) dx \quad \ldots$$

$\boxed{II}$ način

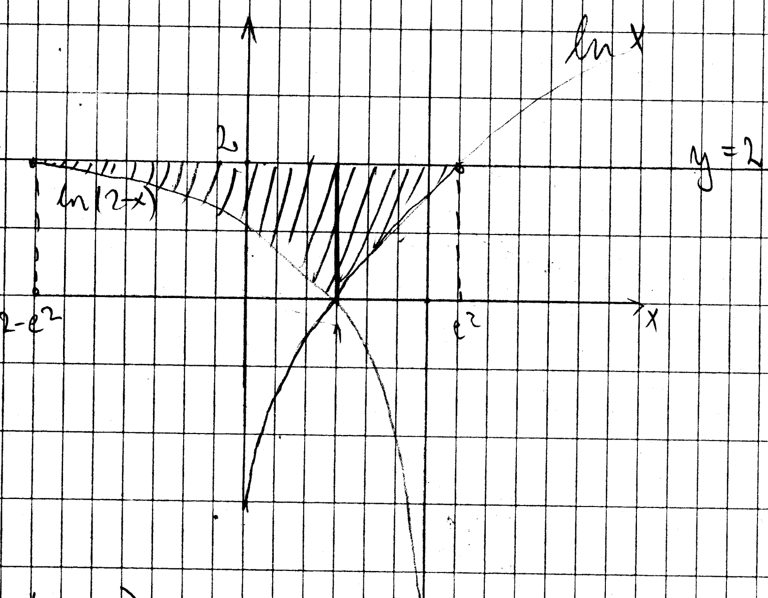
$y^2 = 2x+1$ $\qquad\qquad$ $y = 2x-1$

$y^2 - 1 = 2x$ $\qquad\qquad$ $y+1 = 2x$

$x = \frac{y^2-1}{2}$ $\qquad\qquad$ $x = \frac{y+1}{2}$

$$P = \int_{-1}^{2} \left(\frac{y+1}{2} - \frac{y^2-1}{2} \right) dy = \ldots = \frac{9}{4}$$

b) $y = \ln x$, $y = \ln(2-x)$, $y = 2$



$y = \ln(2-x)$

D.P. $2-x>0$ $\quad x<2$ $\qquad x \in (-\infty, 2)$

nula: $2-x=1$ $\quad \boxed{x=1}$

asimptota:

$\lim\limits_{x\to 2^-} \ln(2-x) = \ln 0_+ = -\infty$ $\qquad$ V.A. $\boxed{x=2}$ (lijeva)

$$\begin{cases} y = \ln x \\ y = 2 \end{cases} \Rightarrow \begin{aligned} \ln x &= 2 \\ x &= e^2 \end{aligned}$$

$$\begin{cases} y = \ln(2-x) \\ y = 2 \end{cases} \qquad \begin{aligned} 2-x &= e^2 \\ x &= 2-e^2 \end{aligned}$$

I način:

dva puta parcijalna integracija

$$P = \int_{2-e^2}^{1} [2 - \ln(2-x)]\,dx + \int_{1}^{e^2} (2 - \ln x)\,dx =$$

II način: (projiciranje po y-osi)

$$\begin{aligned} y &= \ln(2-x) \\ 2-x &= e^y \\ x &= 2-e^y \end{aligned} \qquad \begin{aligned} y &= \ln x \\ x &= e^y \end{aligned}$$

$$P = \int_0^2 [e^y - (2-e^y)]\,dy = \int_0^2 (2e^y - 2)\,dy = (2e^y - 2y)\Big|_0^2$$

$$= 2e^2 - 4 - 2 = 2e^2 - 6$$

za vježbu:

c)* $y = \sin\left(\frac{4}{3}x + \frac{\pi}{6}\right)$, $y = 1$, $x = 0$

d)* $y = \frac{x^2+3}{x+1}$, $y = 3 - x^2$, $y \geq 0$

gledamo samo P iznad x-osi

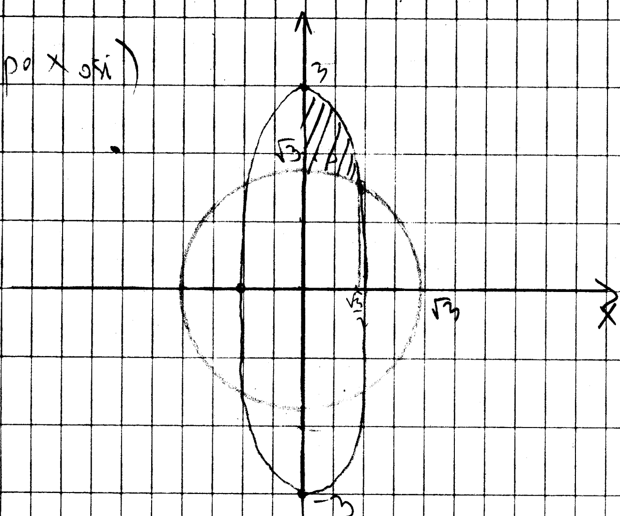
jer se sijeku i iznad i ispod ose

e)* $y = \frac{3}{4}x^2$, $x + y - 5 = 0$, $y = 0$

f) $y^2 = 4 + x$, $y^2 = 3x + 12$

P4) Izračunati površinu dijela ravni koji se nalazi izvan kružnice $x^2 + y^2 = 3$, a unutar elipse $x^2 + \frac{y^2}{9} = 1$ u prvom kvadrantu.

(projiciramo po x osi)



$\frac{x^2}{1} + \frac{y^2}{9} = 1$

$a^2 = 1 \Rightarrow a = 1$

$b^2 = 9 \Rightarrow b = 3$

$\left.\begin{array}{l} x^2 + y^2 = 3 \\ x^2 + \frac{y^2}{9} = 1 \end{array}\right\} -$ $\quad y^2 - \frac{y^2}{9} = 2 \quad / \cdot 9$ $\quad y^2 = \frac{18}{8} = \frac{9}{4}$

$9y^2 - y^2 = 18$ $\quad y = \pm\frac{3}{2}$ $\quad \boxed{y = \frac{3}{2}} \Rightarrow$

$\Rightarrow x^2 + \frac{9}{4} = 3$

$x^2 = 3 - \frac{9}{4} = \frac{3}{4}$ $\quad x^2 = \frac{3}{4}$ $\quad \boxed{x = \pm\frac{\sqrt{3}}{2}}$

$x^2 + y^2 = 3$

$y^2 = 3 - x^2$

$y \geq 0 \Rightarrow y = \sqrt{3 - x^2}$

$x^2 + \frac{y^2}{9} = 1$

$\frac{y^2}{9} = 1 - x^2$

$y^2 = 9(1 - x^2)$

$y \geq 0 \Rightarrow y = 3\sqrt{1 - x^2}$

$$P = \int_0^{\frac{\sqrt{3}}{2}} \left(3\sqrt{1-x^2} - \sqrt{3-x^2}\right) dx = 3\underbrace{\int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\, dx}_{J_1} - \underbrace{\int_0^{\frac{\sqrt{3}}{2}} (3-x^2)\, dx}_{J_2} =$$

$$J_1 = \int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\, dx = \left| \begin{array}{ll} x = \sin t & x = 0 \Rightarrow t = 0 \\ dx = \cos t\, dt & x = \frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{3} \end{array} \right|$$

$$= \int_0^{\frac{\pi}{3}} \underbrace{\sqrt{1-\sin^2 t}}_{\cos^2 t} \cos t\, dt = \int_0^{\frac{\pi}{3}} \cos^2 t\, dt = \int_0^{\frac{\pi}{3}} \frac{1+\cos 2t}{2}\, dt =$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{3}} dt + \int_0^{\frac{\pi}{3}} \cos 2t\, dt \right) = \frac{1}{2}\left(t \Big|_0^{\frac{\pi}{3}} + \frac{1}{2}\sin 2t \Big|_0^{\frac{\pi}{3}} \right) =$$

$$= \frac{1}{2}\left(\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right) = \frac{1}{2}\left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

$$J_2 = \int_0^{\frac{\sqrt{3}}{2}} \sqrt{3-x^2}\, dx = \left| \begin{array}{ll} x = \sqrt{3}\sin t & x = 0 \Rightarrow t = 0 \\ dx = \sqrt{3}\cos t\, dt & x = \frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{6} \end{array} \right|$$

$$= \int_0^{\frac{\pi}{6}} \underbrace{\sqrt{3-3\sin^2 t}}_{3\cos^2 t}\, \sqrt{3}\cos t\, dt = 3\int_0^{\frac{\pi}{6}} \cos^2 t\, dt = 3\cdot\int_0^{\frac{\pi}{6}} \frac{1-\cos 2t}{2}\, dt$$

$$= \frac{3}{2}\left(\int_0^{\frac{\pi}{6}} dt + \int_0^{\frac{\pi}{6}} \cos 2t\, dt \right) = \frac{3}{2}\left(t \Big|_0^{\frac{\pi}{6}} + \frac{1}{2}\sin 2t \Big|_0^{\frac{\pi}{6}} \right)$$

$$= \frac{3}{2}\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{2\pi}{6}\right) = \frac{3\pi}{12} + \frac{3}{4}\cdot\frac{\sqrt{3}}{2} = \frac{\pi}{4} + \frac{3\sqrt{3}}{8}$$

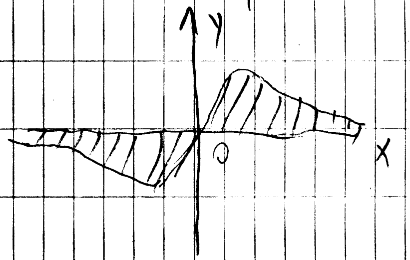
$$P = 3\left(\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right) - \left(\frac{\pi}{4} + \frac{3\sqrt{3}}{8}\right)$$

$$P = \frac{3\pi}{6} + \cancel{\frac{3\sqrt{3}}{8}} - \frac{\pi}{4} - \cancel{\frac{3\sqrt{3}}{8}}$$

$$P = \frac{\pi}{2} - \frac{\pi}{4} \Rightarrow P = \frac{\pi}{4}$$

Za vježbu

① Izračunati P koju obrazuju kriva $y = x e^{-x^2}$ i njena asimptota



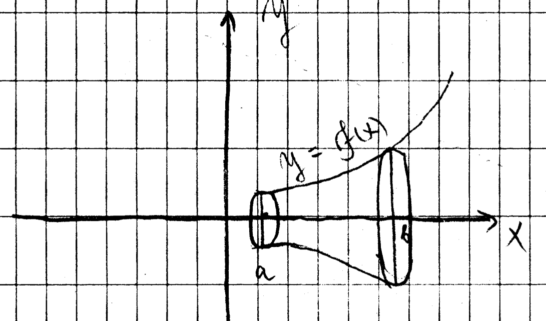
$$P = \int_{-\infty}^{+\infty} x e^{-x^2} dx = 2 \cdot \int_0^{+\infty} x e^{-x^2} dx = \dots$$

② U presječnim tačkama parabole $y = 4 - x^2$ i x-ose povučene su normale na parabolu. Izračunati P figure koja je ograničena parabolom i normalama

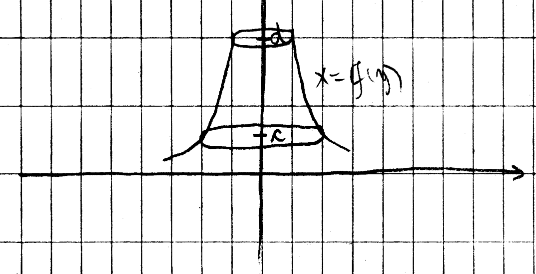
③ Izračunati P elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

27.04.2010.

ZAPREMINA ROTACIONOG TIJELA



$$V_x = \pi \int_a^b [f(x)]^2 dx$$



$$V_y = \pi \cdot \int_c^d [f(y)]^2 dy$$

$$\left.\begin{array}{l} x=\alpha(t) \\ y=\beta(t) \\ t_1 \le t \le t_2 \end{array}\right\} \Rightarrow \quad V_x=\pi \int_{t_1}^{t_2} \left[\beta(t)\right]^2 \cdot |\alpha'(t)| \, dt$$

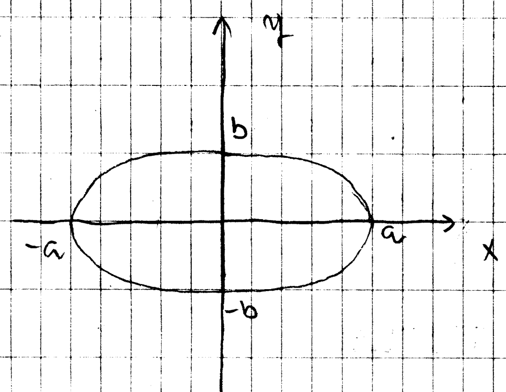
$$V_y=\pi \int_{t_1}^{t_2} \left[\alpha(t)\right]^2 \, |\beta'(t)| \, dt$$

(1.) Izračunati zapreminu tijela koje nastaje rotacijom

a) elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ oko

1° x-ose

2° y-ose

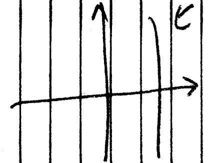


1° $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2} \quad \Big| \cdot b^2 \qquad y^2=\frac{b^2}{a^2}(a^2-x^2)$

$$y^2=\frac{b^2(a^2-x^2)}{a^2}$$

$$V_x=2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx=\frac{2b^2}{a^2}\left(a^2x-\frac{x^3}{3}\right)\Big|_0^a=\frac{2b^2}{a^2}\left(a^3-\frac{a^3}{3}\right)=$$

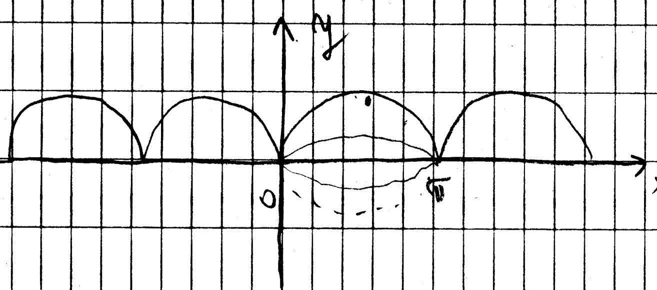
$$=\frac{2b^2}{\cancel{a^2}}\cdot\frac{2a^3}{3}=\frac{4ab^2}{3}\pi$$



2° $x^2 = \frac{a^2}{b^2}(b^2 - y^2)$

$$V_y = 2\pi \int_0^b \frac{a^2}{b^2}(b^2-y^2)\,dy = \dots = \frac{4a^2b}{3}\pi$$

b) $y = \sin^2 x$ oko x-ose, $x \in [0, \pi]$



$y = \sin^2 x$

$$V_x = \pi\int_0^\pi \sin^4 x\,dx = \pi\int_0^\pi \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{\pi}{4}\int_0^\pi (1-2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{\pi}{4}\left[\int_0^\pi dx - \underbrace{2\int_0^\pi \cos 2x\,dx}_{0} + \int_0^\pi \cos^2 2x\,dx\right] =$$

$$= \frac{\pi}{4}\left(\pi + \int_0^\pi \left(\frac{1+\cos 4x}{2}\right)dx\right) = \frac{\pi}{4}\left(\pi + \frac{1}{2}\int_0^\pi dx + \underbrace{\frac{1}{2}\int_0^\pi \cos 4x\,dx}_{0}\right) =$$

$$= \frac{\pi}{4}\cdot\left(\pi + \frac{1}{2}\pi\right) = \frac{\pi}{4}\cdot\frac{3\pi}{2} = \frac{3\pi^2}{8}$$

c) $x = a\cdot(t - \sin t)$ oko x-ose
$y = a\cdot(1-\cos t)$. $0 \le t \le 2\pi$

$$V_x = \pi\cdot\int_0^{2\pi} \underbrace{a^2\cdot(1-\cos t)^2}_{y^2}\cdot\underbrace{a(1-\cos t)}_{|x'|}\,dt$$

$$V_x = a^3\pi \int_0^{2\pi} (1-\cos t)^3 \, dt = a^3\pi \int_0^{2\pi} (1 - 3\cos t + 3\cos^2 t - \cos^3 t)\, dt =$$

$$= a^3\pi \left[\underbrace{\int_0^{2\pi} dt}_{2\pi} - 3 \underbrace{\int_0^{2\pi} \cos t\,dt}_{0} + 3\int_0^{2\pi} \cos^2 t\,dt - \int_0^{2\pi} \cos^3 t\,dt \right] =$$

$$\int_0^{2\pi} \cos^2 t\,dt = \int_0^{\pi} \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\left(\underbrace{\int_0^{2\pi} dt}_{2\pi} + \underbrace{\int_0^{2\pi} \cos 2t\,dt}_{0} \right) = \pi$$

$$\int_0^{2\pi} \cos^3 t\,dt = \int_0^{2\pi} \cos t \cos^2 t\,dt = \int_0^{2\pi} \cos t \cdot (1 - \sin^2 t)\,dt =$$

$$= \left| \begin{array}{ll} \sin t = z & t=0 \Rightarrow z=0 \\ \cos t\,dt = dz & t = 2\pi \Rightarrow z=0 \end{array} \right| = \int_0^0 (1-z^2)\,dz = 0$$

"Integral s istim granicama je uvijek jednak 0".

$$V_x = a^3\pi\,(2\pi + 3\pi) = 5a^3\pi^2$$

d) $y = \sqrt{\dfrac{e^x+2}{(e^x+1)(e^x+3)}}$, oko x-osi.

$x \in \left[0, \frac{\pi}{6}\right]$

e) $y = \arcsin x$ oko x-osi

$x \in (0, 1)$

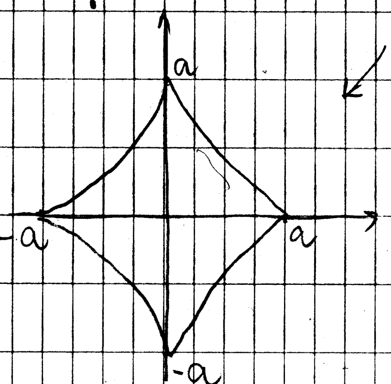
f) $y = e^{2x} \cdot \sin 3x$ oko x-ose

$x \in [0, \ln 2]$

g) $x = a\cos^3 t$ oko y-ose

$y = a \sin^3 t$ $\quad 0 \le t \le 2\pi$

Uputa:



lakše

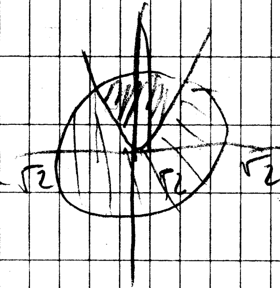
$V_y = \pi \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dots$ ili $= 2\pi \int_{0}^{\frac{\pi}{2}} \dots$

---

4 ZADIT

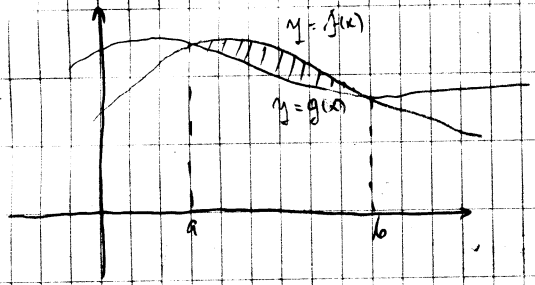
$\int_0^2 \frac{dx}{x^2\sqrt{x^2-1}}$ ← nesvoj. integral

1. Izračunati P 2 dijela kruga $x^2+y^2 \le 2$ koji se dob. u presjeku s parabolom $y = x^2$



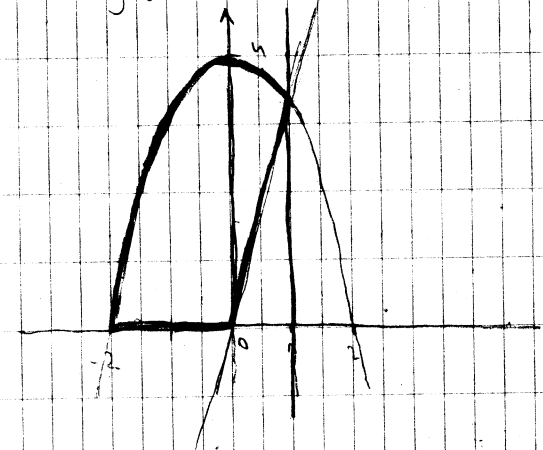
Rj. $P_2 = \frac{3\pi}{2} - \frac{1}{3}$

09.05.2011.



$$V_x = \pi \int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx .$$

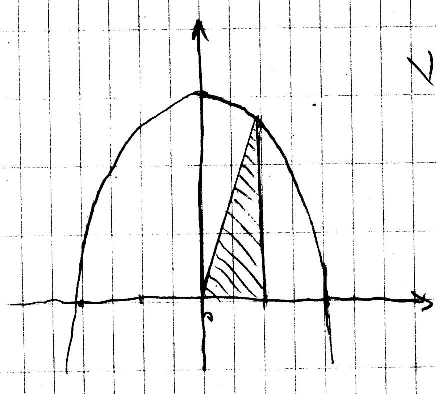
1. Izračunati zapreminu tijela koje nastaje rotacijom oko x-ose figure određene linijama $y=4-x^2$, $y=3x$, $y=0$, $x \le 1$



$V = V_1 - V_2$

$$V_1 = \pi \cdot \int_{-2}^{1} (4-x^2)^2 \, dx = \ldots$$

$$V_2 = \pi \cdot \int_{0}^{1} (3x)^2 \, dx = \ldots$$



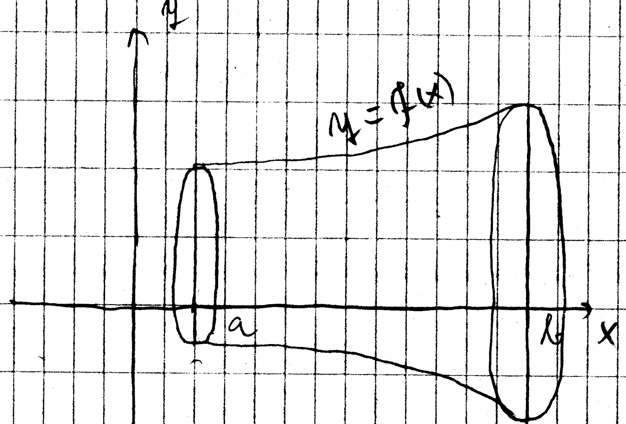
$V = V_1 + V_2$

$$V_1 = \frac{153\pi}{5}$$

$$V_2 = 3\pi$$

$$V = \frac{138\pi}{5}$$

II POVRŠINA OMOTAČA ROTACIONOG TIJELA



$$P_x = 2\pi \cdot \int_a^b |f(x)| \cdot \sqrt{1+(f'(x))^2}\, dx$$

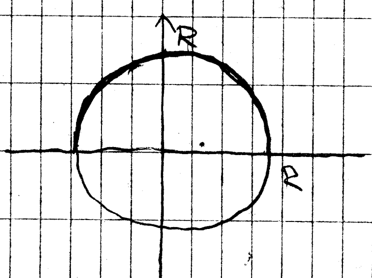
$$P_y = 2\pi \int_a^b |g(y)| \cdot \sqrt{1+(g'(y))^2}\, dy$$

Parametarski oblik.
$$\left.\begin{array}{l} x = \alpha(t) \\ y = \beta(t) \\ t_1 \le t \le t_2 \end{array}\right\} \Rightarrow$$

$$P(x) = 2\pi \cdot \int_{t_1}^{t_2} |\beta(t)| \cdot \sqrt{(\alpha'(t))^2 + (\beta'(t))^2}\, dt$$

$\Rightarrow$
$$P(y) = 2\pi \int_{t_1}^{t_2} |\alpha(t)| \cdot \sqrt{(\alpha'(t))^2 + (\beta'(t))^2}\, dt$$

1. Izračunati površinu lopte poluprečnika r



$x^2 + y^2 = R^2$

$y^2 = R^2 - x^2 \qquad y \ge 0 \Rightarrow y = \sqrt{R^2 - x^2}$

$$y' = \frac{-2x}{2\sqrt{R^2-x^2}} = \frac{-x}{\sqrt{R^2-x^2}}$$

$$P(x) = 2\pi \int_{-R}^{R} \sqrt{R^2-x^2} \cdot \sqrt{1+\frac{x^2}{R^2-x^2}}\, dx$$

$= 2\pi R \int_{-R}^{R} dx = 2\pi R \cdot 2R = 4R^2\pi$

PARAMETRIZACIJA

$\underline{\text{II}}$ način : koristam u Analizi III - KRUŽNICE

$$x^2+y^2=R^2 \iff \begin{cases} x = R\cos t \\ y = R\sin t \\ 0 \le t \le 2\pi \end{cases}$$

$t \in [0,\pi] \Rightarrow$ dobije se 1/2 kružnice

Napomena: parametrizacija elipse ide ovako

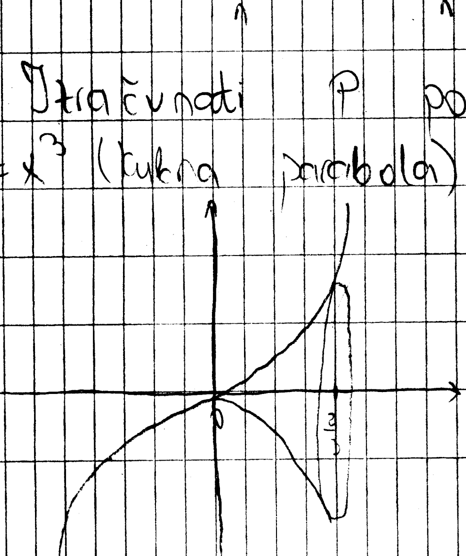
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \begin{cases} x = a\cos t \\ y = b\sin t \\ 0 \le t \le 2\pi \end{cases}$$

$$P(x) = 2\pi \int_0^{\pi} R\sin t \cdot \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$

$$= 2R\pi \int_0^{\pi} \sin t \cdot \sqrt{R^2\underbrace{(\sin^2 t + \cos^2 t)}_{1}}\, dt = 2\pi R^2 (-\cos t)\Big|_0^{\pi}$$

$$= 2\pi R^2 (\underbrace{-\cos\pi}_{1} + \underbrace{\cos 0}_{1}) = 4R^2\pi$$

(2.) Izračunati P površi koja nastaje obrtanjem krive $y=x^3$ (kubna parabola) oko x-ose ako je $0 \le x \le \frac{2}{3}$



$$P_x = 2\pi \int_0^{\frac{2}{3}} \overbrace{|x^3|}^{x>0} \cdot \sqrt{1+(3x^2)^2}\, dx$$

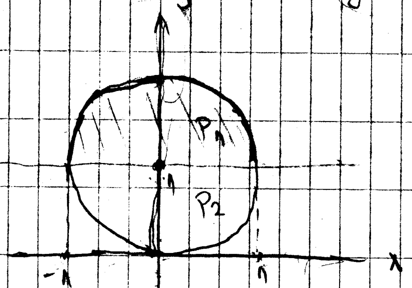
$$= 2\pi \int_0^{2/3} x^3 \cdot \sqrt{1+9x^4}\, dx =$$

$$\left| \begin{array}{ll} 1+9x^4=t & x=0 \Rightarrow t=1 \\ 36x^3dx=dt & x=\frac{2}{3} \Rightarrow t=\frac{25}{9} \\ x^3dx=\frac{1}{36}dt & \end{array} \right| = 2\pi \int_1^{25/9} \sqrt{t}\cdot\frac{dt}{36} =$$

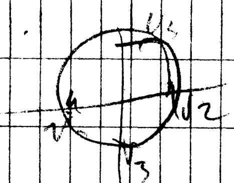
$$= \frac{2\pi}{36}\int_1^{25/9} t^{\frac{1}{2}}\,dt = \frac{\pi}{18}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^{\frac{25}{9}} = \frac{\pi}{18}\cdot\frac{2}{3}\cdot\left(\sqrt{\left(\frac{25}{9}\right)^3}-1\right) =$$

$$= \frac{\pi}{27}\cdot\left(\frac{125}{27}-1\right) = \frac{\pi}{27}\cdot\frac{98}{27} = \frac{98\pi}{729}$$

③ Izračunati P omotača tijela koje nastaje rotacijom kružnice: $x^2+(y-1)^2=1$ oko y-ose.



$x^2+y^2=1$



$$P = P_1 + P_2$$

$$(y-1)^2 = 1-x^2$$

$$y-1 = \pm\sqrt{1-x^2}$$

$$y = 1 \pm \sqrt{1-x^2}$$

$$y = 1 \pm \sqrt{1-x^2} \Rightarrow y' = \pm\frac{-2x}{2\sqrt{1-x^2}} = \mp\frac{x}{\sqrt{1-x^2}}$$

$$P_1 = 2\pi\cdot\int_{-1}^{1}\left(1+\sqrt{1-x^2}\right)\sqrt{1+\frac{x^2}{1-x^2}}\,dx = 2\pi\int_{-1}^{1}\left(1+\sqrt{1-x^2}\right)\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx$$

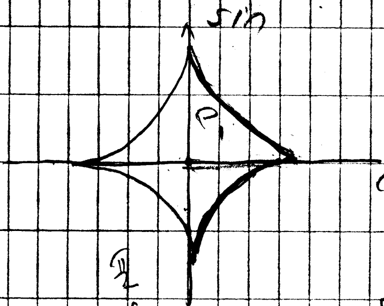
$$= 2\pi\int_{-1}^{1}\frac{1+\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx$$

$$P_2 = 2\pi\int_{-1}^{1}\left(1-\sqrt{1-x^2}\right)\cdot\sqrt{1+\frac{x^2}{1-x^2}}\,dx = 2\pi\int_{-1}^{1}\frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx$$

$$P = 2\pi\int_{-1}^{1}\frac{2}{\sqrt{1-x^2}}\,dx = 4\pi\arcsin x\Big|_{-1}^{1} = 4\pi\cdot\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= 4\pi\cdot\pi = 4\pi^2$$

(4) Izračunati P omotača tijela koje nastaje rotacijom krive $x = a\cos^3 t$, $y = a\sin^3 t$ oko y-ose



$t \in [0, \frac{\pi}{2}] \Rightarrow \frac{1}{2} P_y = \dots$

$$\frac{P_y}{2} = 2\pi \cdot \int_0^{\frac{\pi}{2}} a\cos^3 t \cdot \sqrt{(-3a\cos^2 t \cdot \sin t)^2 + (+3a\sin^2 t \cos t)^2}$$

$$\frac{P_y}{2} = 2\pi \int_0^{\frac{\pi}{2}} a\cos^3 t \cdot \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}\, dt =$$

$$= 2\pi a \int_0^{\frac{\pi}{2}} \cos^3 t \cdot \sqrt{9a^2 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)}\, dt =$$

$$= 2\pi a \int_0^{\frac{\pi}{2}} \cos^3 t \; 3a \sin t \cos t \, dt = 6a^2\pi \int_0^{\frac{\pi}{2}} \sin t \cos^4 t \, dt$$

$$= 6a^2\pi \int_0^{\frac{\pi}{2}} \sin t \, dt \cos^4 t = \left| \begin{array}{lll} \cos t = z & t = 0 & \Rightarrow z = 1 \\ -\sin t\, dt = dz & & \\ \sin t\, dt = -dz & t = \frac{\pi}{2} & \Rightarrow z = 0 \end{array} \right| =$$

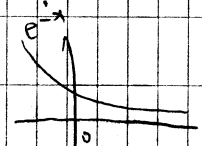
$$= 6a^2\pi \int_1^0 z^4 (-dz) = -6\pi a^2 \cdot \frac{1}{5} = \frac{-6a^2\pi}{5}$$

$$P_y = \frac{12a^2\pi}{5}$$

(5) Izračunati P omotača tijela koje nastaje rotacijom date krive oko x-ose

Dobije se konačna P

a) $y = e^{-x}$, $x \geq 0$ Uputa:



$x \in [0, +\infty]$

b) $y = \sin t$, $x \in [0, \pi]$

c) $x = \frac{(t-2)^2}{4}$, $y = t$, $0 \leq t \leq 4$

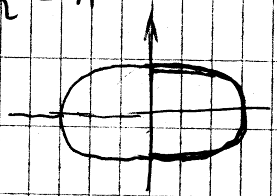
d) $y = \frac{2}{3}\sqrt{x^3}$ $0 \leq x \leq 1$

e) $x=\sqrt{2}\sin t$, $\quad y=\frac{1}{4}\sin 2t$, $\quad 0\le t\le \pi$

f) $x=a\cdot(t\sin t+\cos t)$ $\quad y=a\cdot(\sin t-t\cos t)$, $\quad 0\le t\le \pi$

6. Izračunati P omotača tijela koje nastaje rotacijom elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ oko y-ose!

$(0<b<a)$

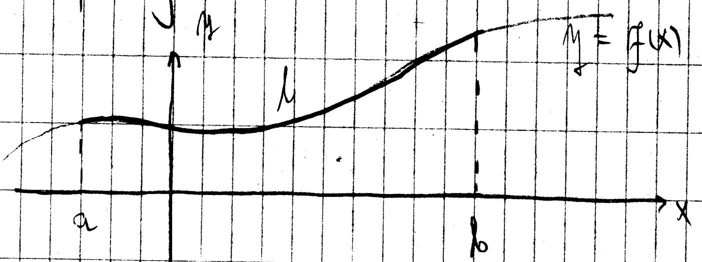


$\frac{1}{2}$ površine

(pokušat s parametrizacijom)

04.05.2010.

IV primjena DUŽINA LUKA KRIVE



$$l=\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$x=f(y) \Rightarrow l=\int_a^b \sqrt{1+(f'(y))^2}\,dy$$

Parametarski: $\left.\begin{array}{l} x=\alpha(t)\\ y=\beta(t)\\ t_1\le t\le t_2\end{array}\right\} \Rightarrow l=\int_{t_1}^{t_2}\sqrt{(\alpha'(t))^2+(\beta'(t))^2}\,dt$

7. Izračunati dužinu luka krive:

a) $y=\frac{x^2}{2}-\frac{\ln x}{4}$, $\quad 1\le x\le 3$, $\quad y'=x-\frac{1}{4x}=\frac{4x^2-1}{4x}$

$$l=\int_1^3\sqrt{1+\left(\frac{4x^2-1}{4x}\right)^2}\,dx=\int_1^3\sqrt{1+\frac{16x^4-8x^2+1}{16x^2}}\,dx$$

$$l = \int_1^3 \sqrt{\frac{16x^2 + 16x^4 - 8x^2 + 1}{16x^2}}\, dx = \int_1^3 \sqrt{\frac{16x^4 + 8x^2 + 1}{16x^2}}\, dx$$

$$l = \int_1^3 \frac{4x^2 + 1}{4x}\, dx = \int_1^3 \frac{4x^2}{4x}\, dx + \int_1^3 \frac{dx}{4x} = \frac{x^2}{2}\Big|_1^3 + \frac{1}{4}\ln x \Big|_1^3 =$$

$$= \frac{9-1}{2} + \frac{1}{4}(\ln 3 - \ln 1) = 4 + \frac{1}{4}\ln 3$$

b) $x = a(\cos t + t\sin t)$
$y = a(\sin t - t\cos t)$
$0 \le t \le \pi$

$$\dot{x} = a\cdot(-\sin t + \sin t + t\cos t) = at\cos t$$
$$\dot{y} = a\cdot(\cos t - \cos t + t\sin t) = at\sin t$$

$$l = \int_0^\pi \sqrt{(at\sin t)^2 + (at\cos t)^2}\, dt = \int_0^\pi \sqrt{a^2t^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^\pi at\, dt = a\cdot\frac{t^2}{2}\Big|_0^\pi = \frac{a\pi^2}{2}$$

c*) $y = \sqrt{x - x^2} + \arcsin\sqrt{x}$, $0 \le x \le 1$

$$y' = \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{1}{2\sqrt{x}} = \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{\sqrt{1-x}\cdot 2\sqrt{x}} =$$

$$= \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{2\sqrt{x-x^2}} = \frac{1-2x+1}{2\sqrt{x-x^2}} = \frac{2(1-x)}{2\sqrt{x-x^2}} = \frac{1-x}{\sqrt{x-x^2}} =$$

$$= \sqrt{\frac{(1-x)^2}{x(1-x)}} = \sqrt{\frac{1-x}{x}}$$

dodamo 1/2 da bi bio tablični

$$l = \int_0^1 \sqrt{1+\frac{1-x}{x}}\,dx = \int_0^1 \sqrt{\frac{x+1-x}{x}}\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \to 0^+} 2\int_\varepsilon^1 \frac{1}{2\sqrt{x}}\,dx$$

$$= \lim_{\varepsilon \to 0^+} 2\sqrt{x}\,\Big|_\varepsilon^1 = \lim_{\varepsilon \to 0}\left(2-\cancel{2\sqrt{\varepsilon}}^{\,0}\right) = 2$$

Za vježbu:

d) $x = \frac{y^2}{4} - \frac{\ln y}{2}, \quad 1 \le y \le e$

e) $y = \sqrt{2x-x^2} - 1, \quad \frac{1}{4} \le x \le 1$

f) $\begin{cases} x = t^2 \\ y = \frac{t^3-3t}{3} \end{cases}$ od tačke $A(0,0)$ do $B(3,0)$

g) $y = \sqrt{1-x^2} + \arcsin x, \quad 0 \le x \le \frac{9}{16}$

h) $x = 3\cdot\left(\cos t + \ln \operatorname{tg}\frac{t}{2}\right) \quad y = 3\sin t \quad \frac{\pi}{2} \le t \le \frac{2\pi}{3}$

V primjena

Cauchy-jev integralni kriterij za pozitivne redove

Red $\sum_{n=1}^{\infty} f(n)$ konvergira ili divergira istovremeno sa nesvojstvenim integralom $\int_1^{+\infty} f(x)dx$ ako su za funkciju $f(x)$ ispunjeni uslovi:

1° $f(x)$ je neprekidna za $x \geq 1$
2° $f(x) \geq 0$ za $x \geq 1$
3° $f(x)$ je nerastuća funkcija $(f'(x) \leq 0)$ za $x \geq 1$

1. Ispitati konvergenciju reda:

a) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ $\qquad f(x) = \frac{\ln x}{x^2}$, $x \geq 2$

1° ispunjen
2° $\ln x \geq 0$ akko $x \geq 1$
3° $y' = \left(\frac{\ln x}{x^2}\right)' = \frac{\frac{1}{x}\cdot x^2 - \ln x \cdot 2x}{x^4} = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$

$1 - 2\ln x < 0 \iff 1 < 2\ln x \iff \ln x > \frac{1}{2} = \ln e^{\frac{1}{2}}$

$\iff x > \sqrt{e}$ $\qquad f'(x) < 0$ ako je $x > \sqrt{e}$, pa $\overset{1.7e}{?}$ $x \geq 2$!

$\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t \to \infty} \int_2^t \frac{\ln x}{x^2} dx = \left| \begin{matrix} u = \ln x & dv = \frac{1}{x^2}dx \\ du = \frac{1}{x}dx & v = -\frac{1}{x} \end{matrix} \right|$

$$= \lim_{t\to\infty}\left(-\frac{1}{x}\ln x\Big|_2^t - \int_2^t \frac{1}{x^2}dx\right) = \lim_{t\to\infty}\left(\underbrace{-\frac{1}{t}\ln t}_{\text{L.P. jer } \frac{\infty}{\infty}} + \frac{\ln 2}{2} - \frac{1}{x}\Big|_2^t\right)$$

$$\overset{\text{L.P.}}{=} \lim_{t\to\infty}\left(-\frac{\cancel{t}}{\cancel{1}}^{\to 0} + \frac{\ln 2}{2} - \frac{1}{\cancel{t}}^{\to 0} + \frac{1}{2}\right) = \frac{1}{2} + \frac{\ln^2}{2} = \frac{1+\ln 2}{2}$$

Rezultat je konačan realan broj $\Rightarrow$ integral konvergira $\Rightarrow$
$\Rightarrow$ red konvergira!
Rezultat integrala nije suma reda!

(2.) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\cdot \ln^{\alpha} n}$ $\quad \alpha > 0$

$$f(x) = \frac{1}{x\cdot \ln^{\alpha} x}, \quad \alpha > 0, \quad x \geq 2$$

$1^\circ, 2^\circ, 3^\circ \Rightarrow$ svi uslovi su očigledno ispunjeni

$\left(3^\circ \; f(x) = \frac{1}{x}\cdot\frac{1}{(\ln x)^{\alpha}} \Rightarrow \ln x \text{ je rastuća} \Rightarrow \frac{1}{\ln x} \text{ je opadajuća}\right)$
$\frac{1}{x} \Rightarrow$ opadajuća funkcija

$$\int_2^{\infty}\frac{1\,dx}{x\cdot\ln^{\alpha}x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\cdot \ln^{\alpha} x} = \left| \begin{matrix} \ln x = z & x=2 \Rightarrow z = \ln 2 \\ \frac{1}{x}dx = dz & x = t \Rightarrow z = \ln t \end{matrix} \right|$$

$$= \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{dz}{z^{\alpha}} = \lim_{t\to+\infty}\int_{\ln 2}^{\ln t} z^{-\alpha}dz = \lim_{t\to+\infty}\frac{z^{1-\alpha}}{1-\alpha}\Big|_{\ln 2}^{\ln t} =$$

$$= \lim_{t\to\infty}\left(\frac{(\ln t)^{1-\alpha}}{1-\alpha} - \frac{(\ln 2)^{1-\alpha}}{1-\alpha}\right) \quad \alpha \neq 1$$

$$\boxed{\ln t \to +\infty \quad (t\to+\infty)}$$

1° $1-\alpha > 0 \iff 1 > \alpha$ red divergira

2° $1-\alpha < 0 \iff 1 - \alpha$ red konvergira

3° $\alpha = 1$ $\displaystyle\int_2^\infty \frac{dx}{x \ln x} = \lim_{t \to +\infty} \int_2^t \frac{dx}{x \ln x} = \left| \begin{matrix} \ln x = z \\ \frac{dx}{x} = dz \end{matrix} \right| =$

$$= \lim_{t \to +\infty} \int_{\ln 2}^{\ln t} \frac{dz}{z} = \lim_{t \to +\infty} \ln z \Big|_{\ln 2}^{\ln t} = \lim_{t \to +\infty} \big(\underbrace{\ln(\ln t)}_{\to\infty} - \ln(\ln 2)\big)$$

$= \infty$

Direktnom provjerom zaključili smo da red divergira za $\alpha = 1$!

Za vježbu:

3. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \ln \frac{n+1}{n-1}$

5. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n \ln n \cdot \ln(\ln n)}$

ZADAĆA: 5 zadataka iz primjena određenih integrala i funkcionalnih nizova

iz različitih lekcija